Indian Statistical Institute
Final Examination 2023-2024
B.Math Third Year
Complex Analysis

Time : 3 Hours Date : 24.04.2024 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Note: (i) Answer all questions. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $U \subseteq \mathbb{C}$ open. (v) $\mathrm{Hol}(U) = \{f : U \to \mathbb{C} \text{ holomorphic }\}$. (vi) $\mathbb{D} = B_1(0)$.

Q1. (14 marks) Prove that there is no entire function f such that

$$f\Big(\frac{1}{n^2}\Big) = \frac{1}{n} \qquad (n \in \mathbb{N}).$$

Q2. (14 marks) Let $f : \mathbb{D} \to \mathbb{C}$ be a function. If $f^2, f^3 \in \mathrm{Hol}(\mathbb{D})$, then prove that

$$f \in \mathrm{Hol}(\mathbb{D}).$$

Q3. (14 marks) Let U be a convex open subset of $\mathbb{C}$, and let $f \in \mathrm{Hol}(U)$. Suppose

$$\mathrm{Real}(f'(z)) > 0 \qquad (z \in U).$$

Prove that f is injective.

Q4. (14 marks) Let $U = \{z \in \mathbb{C} : |z| > 1\}$, and let $f \in \mathrm{Hol}(U)$. If

$$\lim_{z \to \infty} f(z) = 0,$$

then prove that

$$f(\alpha) = \frac{1}{2\pi i} \int_{C_2(0)} \frac{f(z)}{\alpha - z}\, dz,$$

for all $\alpha \in \mathbb{C}$ such that $|\alpha| > 2$.

Q5. (14 marks) Let $n \geq 2$ be a natural number and let $\alpha \in \mathbb{C}$. Prove that

$$\alpha z^n + z + 1,$$

has at least one root in $\overline{B_2(0)}$.

Q6. (14 marks) Use the residue theorem to compute

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^4}\, dx.$$

Q7. (14 marks) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, and assume that $f(\alpha) = \alpha$ and $f(\beta) = \beta$ for two different numbers α and β in $\mathbb{D}$. Prove that

$$f(z) = z \qquad (z \in \mathbb{D}).$$

Q8. (14 marks) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, and assume that $f(\pm\alpha) = 0$ for some $\alpha \in \mathbb{D} \setminus \{0\}$. Prove that

$$|f(0)| \leq |\alpha|^2.$$

If the above is true with equality, then write down the function f explicitly.